

## THE MANY FACES OF THE CHIRAL POTTS MODEL

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In this talk, we give a brief overview of several aspects of the theory of the chiral Potts model, including higher-genus solutions of the star-triangle and tetrahedron equations, cyclic representations of affine quantum groups, basic hypergeometric functions at root of unity, and possible applications.

*Keywords:* Chiral Potts model, star-triangle equation, Yang-Baxter equation, (affine) quantum groups, cyclic representations, basic hypergeometric functions

### 1. Introduction

The chiral Potts model is a generalization of the  $N$ -state Potts model allowing for handedness of the pair interactions. While known under several other names, it has received much interest in the past two decades. It may have appeared first in a short paper by Wu and Wang<sup>1</sup> in the context of duality transformations.

However, active studies of chiral Potts models did not start until a few years after that, when Östlund<sup>2</sup> and Huse<sup>3</sup> introduced the more special chiral clock model as a model for commensurate-incommensurate phase transitions. Much has been published since and in this talk we can only highlight some of the developments and discuss a few of those related to the integrable manifold in more detail.

#### 1.1. *Domain wall theory of incommensurate states in adsorbed layers*

Following the original papers of Östlund<sup>2</sup> and Huse<sup>3</sup> there was immediately much interest in their model, because it can be used to describe wetting phenomena in commensurate phases, transitions to incommensurate states and it provides through the domain wall theory a model for adsorbed monolayers.<sup>4-7</sup>

#### 1.2. *Chiral NN interactions to describe further neighbor interactions*

One may object that next-nearest and further neighbor interactions are to be seen as the physical cause of incommensurate states rather than chiral interactions. However, one can show that by a block-spin transformation such longer-range models can be mapped to a nearest-neighbor interaction model but with chiral

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interactions<sup>8–11</sup> in general. From the viewpoint of integrability, for example, the nearest-neighbor picture is preferred.

### **1.3. *New solutions of quantum Lax pairs, or star–triangle equations***

The integrable chiral Potts model<sup>12–15</sup> provides new solutions of the Yang–Baxter equation, which could help understanding chiral field directions and correlations in lattice and conformal field theories. In fact, the belief that to each conformal field theory corresponds a lattice model was the hidden motivation behind the original discovery.<sup>12</sup>

### **1.4. *Higher-genus spectral parameters (Fermat curves)***

The integrable chiral Potts models are different from all other solvable models based on the Yang–Baxter equations. The spectral parameters (or rapidities) lie on higher-genus curves.<sup>12–15</sup>

### **1.5. *Level crossings in quantum chains (1D vs 2D)***

The physics of incommensurate states in one-dimensional quantum chiral Potts chains is driven by level crossings,<sup>16</sup> which are forbidden in the classical case by the Perron–Frobenius theorem. The two-dimensional chiral Potts model has its integrable submanifold within the commensurate phase, which ends at the Fateev–Zamolodchikov multicritical point.<sup>17</sup>

### **1.6. *Exact solutions for several physical quantities***

All other Yang–Baxter solvable models discovered so far have a uniformization that is based on elementary or elliptic functions, with meromorphic dependences on differences (and sums) of spectral parameters. This is instrumental in the evaluation of their physical quantities. For the integrable chiral Potts model several quantities have been obtained using new approaches without using an explicit uniformization.<sup>16,18–26,7,15</sup>

### **1.7. *Multi-component versions***

Multicomponent versions of the chiral Potts model<sup>27</sup> may be of interest in many fields of study, such as the structure of lipid bilayers for which the Pearce–Scott model has been introduced.<sup>28</sup>

### **1.8. *Large $N$ -limit***

The integrable chiral Potts model allows three large- $N$  limits that may be useful in connection with  $W_\infty$  algebras.<sup>29</sup>

### **1.9. *Cyclic representations of quantum groups at roots of 1***

The chiral Potts model can be viewed as the first application of the theory of cyclic

representations<sup>30</sup> of affine quantum groups.

### 1.10. Generalizing free fermions (*Onsager algebra*)

The operators in the superintegrable subcase of the chiral Potts model obey Onsager’s loop group algebra, making the model integrable for the same two reasons as the  $N = 2$  Ising or free-fermion model. Howes, Kadanoff, and den Nijs<sup>31</sup> first noted special features of series expansions for a special case of the  $N = 3$  quantum chain. This was generalized by von Gehlen and Rittenberg<sup>32</sup> to arbitrary  $N$  using the Dolan–Grady criterium, which was later explained to be equivalent to Onsager’s loop algebra relations.<sup>33,34</sup>

### 1.11. Solutions of tetrahedron equations

Bazhanov and Baxter<sup>35–43</sup> have shown that the  $\mathfrak{sl}(n)$  generalization<sup>44–47</sup> of the integrable chiral Potts model can be viewed as an  $n$ -layered  $N$ -state generalization of the three-dimensional Zamolodchikov model.

### 1.12. Cyclic hypergeometric functions

Related is a new theory of basic hypergeometric functions at root of unity discussed in some detail at the end of this talk.

### 1.13. New models with few parameters

In the integrable submanifold with positive Boltzmann weights several ratios of the parameters are nearly constant, suggesting the study of new two-parameter  $N$ -state models.<sup>7,48</sup>

## 2. The Chiral Potts Model

The most general chiral Potts model is defined on a graph or lattice, see Fig. 1, where the interaction energies

$$\mathcal{E}(n) = \mathcal{E}(n + N) = \sum_{j=1}^{N-1} E_j \omega^{jn}, \quad \omega \equiv e^{2\pi i/N}, \quad (1)$$

depend on the differences  $n = a - b \bmod N$  of the spin variables  $a$  and  $b$  on two neighboring sites. We can write

$$\frac{E_j}{k_B T} = \frac{E_{N-j}^*}{k_B T} = -K_j \omega^{\Delta_j}, \quad j = 1, \dots, \lfloor \tfrac{1}{2}N \rfloor, \quad (2)$$

where  $K_j$  and  $\Delta_j$  constitute  $N - 1$  independent variables. Then, for  $N$  odd we have a sum of “clock model” terms

$$-\frac{\mathcal{E}(n)}{k_B T} = \sum_{j=1}^{\frac{1}{2}(N-1)} 2K_j \cos \left[ \frac{2\pi}{N}(jn + \Delta_j) \right], \quad (3)$$

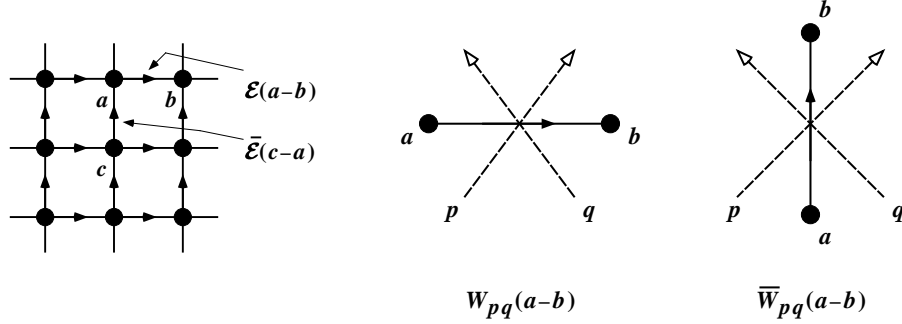


Fig. 1. Interaction energies and Boltzmann weights of horizontal and vertical couplings in the chiral Potts model. The subtraction of the state variables  $a, b, c = 1, 2, \dots, N$  at the lattice sites is done modulo  $N$ . The dashed lines with open arrows denote oriented lines on the medial lattice where the rapidity variables  $p, q, \dots$ , of the integrable submanifold live.

whereas, for  $N$  even we have an additional Ising term, *i.e.*

$$-\frac{\mathcal{E}(n)}{k_B T} = \sum_{j=1}^{\frac{1}{2}N-1} 2K_j \cos \left[ \frac{2\pi}{N}(jn + \Delta_j) \right] + K_{\frac{1}{2}N}(-1)^n. \quad (4)$$

The Boltzmann weight corresponding to the edge is given by

$$W(n) = e^{-\mathcal{E}(n)/k_B T}. \quad (5)$$

### 3. The Integrable Chiral Potts Model

In the integrable chiral Potts model, we have besides “spins”  $a, b, \dots$  defined mod  $N$ , “rapidity lines”  $p, q, \dots$  all pointing in one halfplane. The weights satisfy the star-triangle equation, see also Fig. 2,

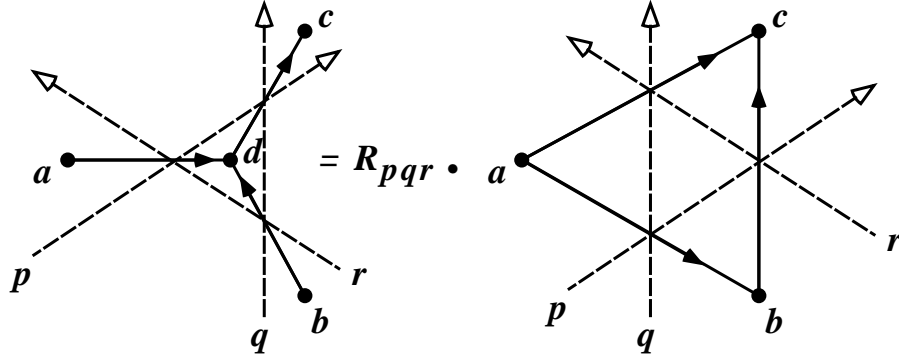


Fig. 2. Star-Triangle Equation.

$$\sum_{d=1}^N \overline{W}_{qr}(b-d)W_{pr}(a-d)\overline{W}_{pq}(d-c) = R_{pqr}W_{pq}(a-b)\overline{W}_{pr}(b-c)W_{qr}(a-c). \quad (6)$$

In full generality there are six sets of weights to be found and a constant  $R$ . But the solution is in terms of two functions depending on spin differences and pairs of rapidity variables:

$$\begin{aligned} W_{pq}(n) &= W_{pq}(0) \prod_{j=1}^n \left( \frac{\mu_p}{\mu_q} \cdot \frac{y_q - x_p \omega^j}{y_p - x_q \omega^j} \right), \\ \overline{W}_{pq}(n) &= \overline{W}_{pq}(0) \prod_{j=1}^n \left( \mu_p \mu_q \cdot \frac{\omega x_p - x_q \omega^j}{y_q - y_p \omega^j} \right), \end{aligned} \quad (7)$$

with  $R$  depending on three rapidity variables. Periodicity modulo  $N$  gives for all rapidity pairs  $p$  and  $q$

$$\left( \frac{\mu_p}{\mu_q} \right)^N = \frac{y_p^N - x_q^N}{y_q^N - x_p^N}, \quad (\mu_p \mu_q)^N = \frac{y_q^N - y_p^N}{x_p^N - x_q^N}. \quad (8)$$

Hence, we can define (rescale)  $k, k'$  such that

$$\mu_p^N = \frac{k'}{1 - k x_p^N} = \frac{1 - k y_p^N}{k'}, \quad x_p^N + y_p^N = k(1 + x_p^N y_p^N), \quad k^2 + k'^2 = 1, \quad (9)$$

and similarly with  $p$  replaced by  $q$ . The rapidities live on a curve of genus  $g > 1$  that is of Fermat type.

#### 4. Physical Cases

There are two conditions for physical cases:

I. Planar Model with Real Positive Boltzmann Weights,

II. Hermitian Quantum Spin Chain.

Usually, as in the six-vertex model in electric fields where the quantum chain would have either imaginary or real Dzyaloshinsky–Moriya interactions, one cannot require both simultaneously. Only for the nonchiral (reflection-positive) subcase of the Fateev–Zamolodchikov model<sup>17</sup> are both physical conditions simultaneously fulfilled for  $N > 2$ .

We note that the Hermitian quantum spin chain submanifold contains the superintegrable case, where both the star–triangle (or Yang–Baxter) equation and the Onsager algebra are satisfied.

#### 5. Generalization of Free Fermion Model

Combining four chiral Potts model weights in a square, as in Fig. 3, we obtain the

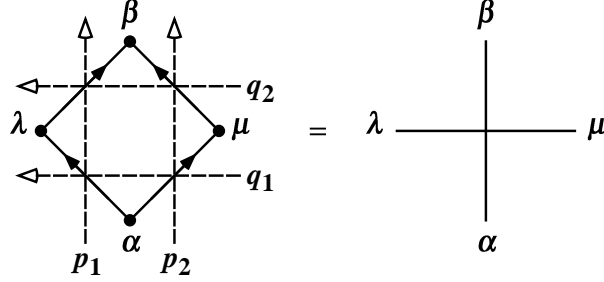


Fig. 3. Vertex model satisfying the Yang-Baxter Equation.

R-matrix of the  $N$ -state generalization<sup>14</sup> of the checkerboard Ising model

$$R_{\alpha\beta|\lambda\mu} = \overline{W}_{p_1q_1}(\alpha - \lambda)\overline{W}_{p_2q_2}(\mu - \beta)W_{p_2q_1}(\alpha - \mu)W_{p_1q_2}(\lambda - \beta). \quad (10)$$

Applying a Fourier transform gauge transformation we obtain<sup>14</sup> from this

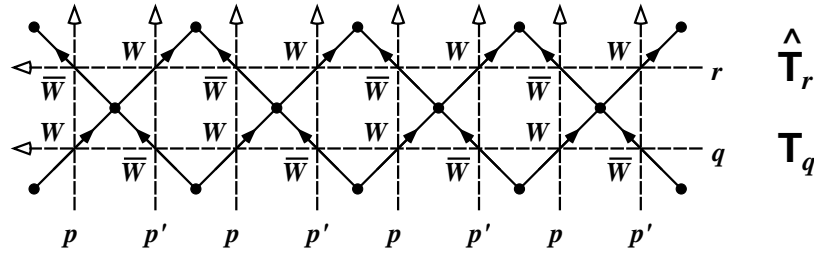
$$\hat{R}_{\alpha\beta|\lambda\mu} = \frac{s_\beta t_\lambda}{s_\alpha t_\mu} \frac{1}{N^2} \sum_{\alpha'=1}^N \sum_{\beta'=1}^N \sum_{\lambda'=1}^N \sum_{\mu'=1}^N \omega^{-\alpha\alpha' + \beta\beta' + \lambda\lambda' - \mu\mu'} R_{\alpha'\beta'|\lambda'\mu'}, \quad (11)$$

which is the R-matrix of an  $N$ -state generalization of the free-fermion eight-vertex model for  $N = 2$ . Indeed,  $\hat{R}_{\alpha\beta|\lambda\mu}$  is nonzero only if  $\lambda + \beta = \alpha + \mu \bmod N$  which generalizes the eight-vertex condition.

In eq. (11)  $s_\alpha$  and  $t_\lambda$  are free parameters, corresponding to gauge freedom, which may be edge dependent.

## 6. Transfer Matrices

If the weight functions  $W$  and  $\overline{W}$  are solutions of the star-triangle equation (6)

Fig. 4. Diagonal-to-diagonal transfer matrices  $T_q$  and  $\hat{T}_r$ .

and parametrized as in (7) with variables on rapidity lines lying on the algebraic curve (9), all diagonal transfer matrices commute. This is depicted in Fig. 4, where the vertical rapidity variables have been chosen alternatingly; more generally they could all be chosen independently.

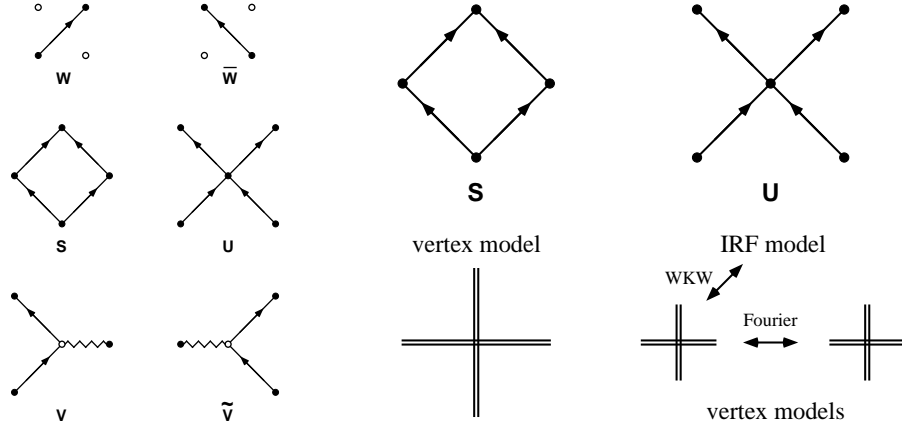


Fig. 5. Various R-matrices related to the chiral Potts model. Vertex weight  $S$  and IRF weight  $U$  are built from two  $W$ 's and two  $\bar{W}$ 's and have two horizontal and two vertical rapidity lines. By the Wu–Kadanoff–Wegner map, IRF weight  $U$  is also a vertex weight. The two zigzag lines in  $V$  and  $\tilde{V}$  represent Fourier and inverse Fourier transform, respectively.

We note that there are several ways to introduce R-matrices, as is indicated in Fig. 5. First, we have the original weights  $W$  and  $\bar{W}$ . Next, we have the vertex weight (square)  $S$  and the IRF weight (star)  $U$ , both consisting of two  $W$  and two  $\bar{W}$  weights. Finally, we have the three-spin interactions  $V$  and  $\tilde{V}$ , its transposed or rotated version, which play a special role in the theory.<sup>12,15</sup> Note that  $S$  and  $U$  can be constructed from a  $V$  and a  $\tilde{V}$ , for example as  $U = V \cdot \tilde{V}$ .

## 7. The Construction of Bazhanov and Stroganov

Bazhanov and Stroganov<sup>49</sup> have shown that the R-matrix  $S$  of a square in the checkerboard chiral Potts model,<sup>14</sup> see Figs. 3 and 5, is the intertwiner of two cyclic representations. Their procedure is sketched in Fig. 6.

They start from the six-vertex model R-matrix at an  $N$ th root of 1, with  $N$  odd. The corresponding Yang–Baxter equation is well-known with spin  $\frac{1}{2}$  highest-weight representations on all legs. They then look for an R-matrix  $L$  intertwining a highest-weight and a cyclic representation. This is solved from a new Yang–Baxter equation that is quadratic in  $L$ .<sup>a</sup> Next, the Yang–Baxter equation with one highest-weight and two cyclic rapidity lines is a linear equation for the intertwiner  $S$  of two cyclic representations. This intertwiner  $S$  satisfies a Yang–Baxter equation with only cyclic representations on the legs. The original result<sup>14</sup> for  $S$  is obtained in a suitable gauge with a proper choice of parameters.

The above illustrates the group theoretical significance of the chiral Potts model. It gives a standard example of the affine quantum group  $U_q(\widehat{\mathfrak{sl}(2)})$  at root of unity in the minimal cyclic representation.<sup>30</sup> Intertwiners of certain bigger irreducible cyclic

<sup>a</sup>Korepanov had earlier obtained the first part of this construction, but his work has not yet been published.<sup>50</sup>

representations are given by chiral Potts model partition functions, as was found by Tarasov.<sup>51</sup>

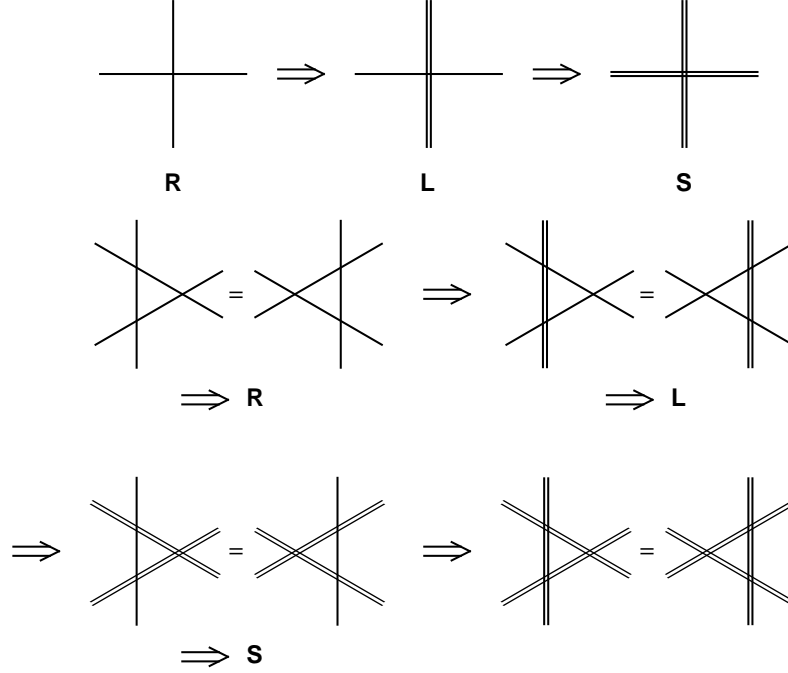


Fig. 6. The construction of Bazhanov and Stroganov. Single lines correspond to spin  $\frac{1}{2}$  highest-weight representations; double lines correspond to minimal cyclic representations.

Smaller representations (*i.e.* semicyclic<sup>52–54</sup> and highest-weight) follow by the reduction of products of cyclic representations in the construction of Baxter *et al.*<sup>24</sup> Elsewhere we shall present more details on how this relates several root-of-unity models. Relating  $p'$  in Fig. 4 by a special automorphism with  $p$ , the product of transfer matrices splits as<sup>24</sup>

$$\mathsf{T}_q \hat{\mathsf{T}}_r = B_{pp'q}^{(j)} \mathsf{X}^{-k} \tau^{(j)}(t_q) + B_{p'pq}^{(N-j)} \mathsf{X}^l \tau^{(N-j)}(t_r). \quad (12)$$

Here the transfer matrix  $\tau^{(j)}$  is made up of L-operators intertwining a cyclic and a spin  $s = \frac{j-1}{2}$  representation, the  $B$ 's are scalars,  $t_q \equiv x_q y_q$ , and powers of the spin shift operator  $\mathsf{X}$  come in depending on the automorphism.

Doing the same process  $r = q'$  in the other direction, one obtains several fusion relations for the  $\tau^{(j)}$  transfer matrices<sup>24</sup>

$$\begin{aligned} \tau^{(j)}(t_q) \tau^{(2)}(\omega^{j-1} t_q) &= z(\omega^{j-1} t_q) \tau^{(j-1)}(t_q) + \tau^{(j+1)}(t_q), \\ \tau^{(j)}(\omega t_q) \tau^{(2)}(t_q) &= z(\omega t_q) \tau^{(j-1)}(\omega^2 t_q) + \tau^{(j+1)}(t_q), \end{aligned} \quad (13)$$

where  $z(t)$  is a known scalar function.

## 8. Selected Exact Results

There are several exact results for the free energy of the integrable chiral Potts model and we quote here a recent result of Baxter in the scaling regime,<sup>23</sup>

$$\begin{aligned} f - f_{\text{FZ}} &\equiv f - f_c = -\frac{(N-1)k^2}{2N\pi}(u_q - u_p)\cos(u_p + u_q) \\ &+ \frac{k^2}{4\pi^2}\sin(u_q - u_p)\sum_{j=1}^{<N/2}\frac{\tan(\pi j/N)}{j}B\left(1 + \frac{j}{N}, \frac{1}{2}\right)^2\left(\frac{k}{2}\right)^{4j/N} \\ &+ O(k^4 \log k), \quad \text{if } k^2 \sim T_c - T \rightarrow 0, \quad \alpha = 1 - \frac{2}{N}. \end{aligned} \quad (14)$$

For the order parameters we have a general conjecture<sup>18</sup> in the ordered state,

$$\langle \sigma_0^n \rangle = (1 - k'^2)^{\beta_n}, \quad \beta_n = \frac{n(N-n)}{2N^2}, \quad (1 \leq n \leq N-1, \quad \sigma_0^N = 1), \quad (15)$$

which still remains to be proved.

Using Baxter's results<sup>25</sup> we have a very explicit formula<sup>7</sup> for the interfacial tensions,

$$\frac{\epsilon_r}{k_B T} = \frac{8}{\pi} \int_0^\eta dy \frac{\sin(\pi r/N)}{1 + 2y \cos(\pi r/N) + y^2} \operatorname{artanh} \sqrt{\frac{\eta^N - y^N}{1 - \eta^N y^N}}, \quad (16)$$

in the fully symmetric,  $W \equiv \overline{W}$ , integrable chiral Potts model, *i.e.* diagonal chiral fields. Here  $\eta \equiv [(1 - k')/(1 + k')]^{1/N}$  is a temperature-like variable and  $r$  is the difference of the spin variables across the interface, ( $r = 1, \dots, N-1$ ).

In the low-temperature region we have  $k' \rightarrow 0$  and we can expand (16) as<sup>7</sup>

$$\frac{\epsilon_r}{k_B T} = -\frac{2r}{N} \log \frac{k'}{2} - \log \left( \frac{1}{\cos^2 r \bar{\lambda}} \prod_{j=1}^{[r/2]} \frac{\cos^4(r - 2j + 1)\bar{\lambda}}{\cos^4(r - 2j)\bar{\lambda}} \right) + O(k'), \quad (17)$$

where  $\bar{\lambda} \equiv \pi/(2N)$  and the constant term comes from a dilogarithm integral. For the critical region,  $\eta \approx (k/2)^{2/N} \sim (T_c - T)^{1/N} \rightarrow 0$ , (16) reduces to<sup>7</sup>

$$\begin{aligned} \frac{\epsilon_r}{k_B T} &= \frac{8 \sin(\pi r/N) B(1/N, 1/2)}{\pi(N+2)} \eta^{1+N/2} \\ &- \frac{8 \sin(2\pi r/N) B(2/N, 1/2)}{\pi(N+4)} \eta^{2+N/2} + O(\eta^{3+N/2}) \\ &\approx \eta^{N\mu} D_r(\eta) = \eta^{N\mu} D_r(\Delta/\eta^{N\phi}), \end{aligned} \quad (18)$$

where we have assumed the existence of scaling function  $D_r$  depending on  $\eta$  and  $\Delta \sim (T_c - T)^{1/2}$ , the chiral field strength on the integrable line. From this, we have the critical exponents

$$\mu = \frac{1}{2} + \frac{1}{N} = \nu, \quad \phi = \frac{1}{2} - \frac{1}{N}. \quad (19)$$

We note that the above dilogarithm identity involves the dilogarithms at  $2N$ -th roots of unity and has been discovered numerically first. Apart from direct proofs for small  $N = 2, 3, 4$ , only an indirect proof<sup>7</sup> by the Bethe Ansatz exists.

### 9. Basic Hypergeometric Series at Root of Unity

The basic hypergeometric series is defined<sup>55,56</sup> as

$${}_{p+1}\Phi_p \left[ \begin{matrix} \alpha_1, \dots, \alpha_{p+1} \\ \beta_1, \dots, \beta_p \end{matrix} ; z \right] = \sum_{l=0}^{\infty} \frac{(\alpha_1; q)_l \cdots (\alpha_{p+1}; q)_l}{(\beta_1; q)_l \cdots (\beta_p; q)_l (q; q)_l} z^l, \quad (20)$$

where

$$(x; q)_l \equiv \begin{cases} 1, & l = 0, \\ (1-x)(1-xq) \cdots (1-xq^{l-1}), & l > 0, \\ 1/[(1-xq^{-1})(1-xq^{-2}) \cdots (1-xq^l)], & l < 0. \end{cases} \quad (21)$$

Setting  $\alpha_{p+1} = q^{1-N}$  and  $q \rightarrow \omega \equiv e^{2\pi i/N}$ , we get

$${}_{p+1}\Phi_p \left[ \begin{matrix} \omega, \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_p \end{matrix} ; z \right] = \sum_{l=0}^{N-1} \frac{(\alpha_1; \omega)_l \cdots (\alpha_p; \omega)_l}{(\beta_1; \omega)_l \cdots (\beta_p; \omega)_l} z^l. \quad (22)$$

We note

$$(x; \omega)_{l+N} = (1-x^N)(x; \omega)_l, \quad \text{and} \quad (\omega; \omega)_l = 0, \quad l \geq N. \quad (23)$$

So if we also require

$$z^N = \prod_{j=1}^p \gamma_j^N, \quad \gamma_j^N = \frac{1 - \beta_j^N}{1 - \alpha_j^N}, \quad (24)$$

we obtain a “cyclic basic hypergeometric function” with summand periodic mod  $N$ .

For us the Saalschütz case, defined by

$$z = q = \frac{\beta_1 \cdots \beta_p}{\alpha_1 \cdots \alpha_{p+1}} \quad \text{or} \quad \omega^2 \alpha_1 \alpha_2 \cdots \alpha_p = \beta_1 \beta_2 \cdots \beta_p, \quad z = \omega, \quad (25)$$

is important, but details on this will be given elsewhere.

The theory of cyclic hypergeometric series is intimately related with the theory of the integrable chiral Potts model and several identities appear hidden in the literature. We note that our notations here, which match up nicely with the classical definitions of basic hypergeometric functions,<sup>55,56</sup> differ from those of Bazhanov *et al.*,<sup>35–43</sup> who are using an upside-down version of the  $q$ -Pochhammer symbol  $(x; q)_l$ . Hence, our definition of the cyclic hypergeometric series differs from the one of Sergeev *et al.*,<sup>43</sup> who also use homogeneous rather than more compact affine variables. So comparing our results with theirs is a little cumbersome.

### 9.1. Integrable chiral Potts model weights

The weights of the integrable chiral Potts model can be written in product form

$$\frac{W(n)}{W(0)} = \gamma^n \frac{(\alpha; \omega)_n}{(\beta; \omega)_n}, \quad \gamma^N = \frac{1 - \beta^N}{1 - \alpha^N}. \quad (26)$$

This is periodic with period  $N$  as follows from (23).

The dual weights are given by Fourier transform,<sup>1</sup> *i.e.*

$$W^{(f)}(k) = \sum_{n=0}^{N-1} \omega^{nk} W(n) = {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \omega^k \right] W(0). \quad (27)$$

Using the recursion formula

$$W(n)(1 - \beta \omega^{n-1}) = W(n-1) \gamma (1 - \alpha \omega^{n-1}) \quad (28)$$

and its Fourier transform, we find

$$\frac{W^{(f)}(k)}{W^{(f)}(0)} = \frac{{}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \omega^k \right]}{{}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \right]} = \left( \frac{\omega}{\beta} \right)^k \frac{(\gamma; \omega)_k}{(\omega \alpha \gamma / \beta; \omega)_k}. \quad (29)$$

This relation is equivalent to the one originally found in 1987,<sup>14,15</sup> It shows that dual weights also satisfy (26).

With one more Fourier transform we get

$$\frac{N}{{}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \right]} = {}_2\Phi_1 \left[ \begin{matrix} \omega, \gamma \\ \omega \alpha \gamma / \beta, \frac{\omega}{\beta} \end{matrix} \right] = {}_2\Phi_1 \left[ \begin{matrix} \omega, \beta / \alpha \gamma \\ \omega / \gamma \end{matrix}; \alpha \right]. \quad (30)$$

Also, we can show that

$$\frac{{}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \omega^m \\ \beta \omega^n \end{matrix}; \gamma \omega^k \right]}{{}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \right]} = \frac{(\omega / \beta)^k (\beta; \omega)_n (\gamma; \omega)_k (\omega \alpha / \beta; \omega)_{m-n}}{(\gamma \omega^k)^n (\alpha; \omega)_m (\omega \alpha \gamma / \beta; \omega)_{m-n+k}}. \quad (31)$$

This equation has been proved by Kashaev *et al.*<sup>36</sup> in other notation and is valid for all values of the arguments, provided condition (26) holds.

### 9.2. Baxter's summation formula

From Baxter's work<sup>20</sup> we can infer the identity

$$\begin{aligned} {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \right] &= \Phi_0 \sqrt{N} (\omega / \beta)^{\frac{1}{2}(N-1)} \\ &\times \prod_{j=1}^{N-1} \left[ \frac{(1 - \omega^{j+1} \alpha / \beta)(1 - \omega^j \gamma)}{(1 - \omega^j \alpha)(1 - \omega^{j+1} / \beta)(1 - \omega^{j+1} \alpha \gamma / \beta)} \right]^{j/N}, \end{aligned} \quad (32)$$

valid up to an  $N$ -th root of unity, while

$$\Phi_0 \equiv e^{i\pi(N-1)(N-2)/12N}, \quad \gamma^N = \frac{1 - \beta^N}{1 - \alpha^N}. \quad (33)$$

Introducing a function

$$p(\alpha) = \prod_{j=1}^{N-1} (1 - \omega^j \alpha)^{j/N}, \quad p(0) = 1, \quad (34)$$

we can rewrite the identity as

$${}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha \\ \beta \end{matrix}; \gamma \right] = \omega^d N^{\frac{1}{2}} \Phi_0 \left( \frac{\omega}{\beta} \right)^{\frac{1}{2}(N-1)} \frac{p(\omega\alpha/\beta)p(\gamma)}{p(\alpha)p(\omega/\beta)p(\omega\alpha\gamma/\beta)}, \quad (35)$$

with  $\omega^d$  determined by the choice of the branches. The LHS of (35) is single valued in  $\alpha$ ,  $\beta$ , and  $\gamma$ , whereas the RHS has branch cuts. It is possible to give a precise prescription for  $d$ , assuming that  $p(\alpha)$  has branch cuts along  $[\omega^j, \infty)$  for  $j = 1, \dots, N-1$ , following straight lines through the origin; but these details will be presented elsewhere.

Using this relation (35) and classical identities we can get many relations for  ${}_3\Phi_2$ ,  ${}_4\Phi_3$ ,  ${}_5\Phi_4, \dots$ , including the star-triangle equation and the tetrahedron equation of the Bazhanov–Baxter model. Some of these results have been given very recently in different notation.<sup>43</sup>

### 9.3. Outline of proof of Baxter's identity

As (35) is crucial in the theory, we briefly sketch a proof here. Also we note that in this subsection, we choose the normalization  $W(0) = 1$  for the weights given by (26), rather than  $\prod W(j) = 1$ .

We note that the  $W^{(f)}(k)$  are eigenvalues of the cyclic  $N \times N$  matrix

$$\mathbf{M} \equiv \begin{pmatrix} W(0) & W(1) & \dots & W(N-1) \\ W(-1) & W(0) & \dots & W(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ W(1-N) & W(2-N) & \dots & W(0) \end{pmatrix}, \quad (36)$$

and

$$\det \mathbf{M} = \prod_{j=0}^{N-1} W^{(f)}(j) = [W(0)^{(f)}]^N \prod_{j=1}^{N-1} \left[ \frac{W^{(f)}(j)}{W^{(f)}(0)} \right]. \quad (37)$$

So if we can calculate  $\det \mathbf{M}$  directly, we obtain a result for  $[W(0)^{(f)}]^N$ , giving us the proof of the identity.

Using

$$W(n) = \gamma^n \prod_{j=0}^{n-1} \frac{1 - \alpha\omega^j}{1 - \beta\omega^j}, \quad W(N-n) = W(-n) = \gamma^{-n} \prod_{j=-n}^{-1} \frac{1 - \beta\omega^j}{1 - \alpha\omega^j}, \quad (38)$$

we can rewrite

$$\det \mathbf{M} = \prod_{l=0}^{N-1} \left[ \prod_{j=0}^{N-2-l} (1 - \omega^j \beta)^{-1} \prod_{j=-l}^{-1} (1 - \omega^j \alpha)^{-1} \right] \det \mathbf{E}^{(0)}, \quad (39)$$

where the elements of  $\mathbf{E}^{(0)}$  are polynomials, so that  $\det \mathbf{E}^{(0)}$  is also a polynomial. We can define more general matrix elements  $\mathbf{E}^{(m)}$  for  $m = 0, \dots, N-1$  by

$$E_{k,l}^{(m)} = \prod_{j=-k+1}^{l-k-1} (1 - \omega^j \alpha) \prod_{j=l+m-k}^{N-k-1} (1 - \omega^j \beta), \quad (40)$$

which satisfy the recursion relation

$$E_{k,l}^{(m)} - E_{k,l+1}^{(m)} = \omega^{l-k} (\alpha - \omega^m \beta) E_{k,l}^{(m+1)}. \quad (41)$$

Subtracting the pairs of consecutive columns of  $\det \mathbf{E}^{(0)}$  in (39), and using (41), we can pull out some of the zeros leaving a determinant with  $N-1$  columns from  $\mathbf{E}^{(1)}$  and the last column from  $\mathbf{E}^{(0)}$ . Repeating this process, we arrive at

$$\det \mathbf{E}^{(0)} = \prod_{m=0}^{N-2} (\alpha - \omega^m \beta)^{N-1-m} \cdot \det \mathbf{F}. \quad (42)$$

Here the matrix  $\mathbf{F}$  is defined such that its  $j$ th column is the  $j$ th column of matrix  $\mathbf{E}^{(N-j)}$ .

From a simple polynomial degree count we conclude that  $\det \mathbf{F}$  has to be a constant. Noting that  $\mathbf{E}^{(0)}$  is triangular in the limit  $\alpha \rightarrow 1$ , we find

$$\det \mathbf{F} = \prod_{j=1}^{N-1} (1 - \omega^j)^j = \Phi_0^N N^{\frac{1}{2}N}, \quad (43)$$

in which  $\Phi_0$  is given by (33). Hence, we can complete the proof of the identity.

#### 9.4. Further identities

Several other identities can be derived using the above identity and the classical Jackson identity.<sup>55,56</sup> One thus generates the fundamental identities for the weights of the Baxter–Bazhanov model and the  $\mathfrak{sl}(n)$  chiral Potts model. More precisely, the Boltzmann weights of a cube in the Baxter–Bazhanov model are proportional to  ${}_3\Phi_2$ 's, so all identities for the weights of this model are also identities for cyclic  ${}_3\Phi_2$ 's.

Without any restriction on the parameters, we can derive

$${}_3\Phi_2 \left[ \begin{matrix} \omega, \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{matrix}; z \right] = N^{-1} \sum_{k=0}^{N-1} {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha_1 \\ \beta_1 \end{matrix}; \omega^{-k} \gamma_1 \right] {}_2\Phi_1 \left[ \begin{matrix} \omega, \alpha_2 \\ \beta_2 \end{matrix}; \omega^k \gamma_2 \right], \quad (44)$$

where  $z = \gamma_1 \gamma_2$  and  $\gamma_i$  is defined in (24). We have only used the convolution theorem so far. Next, we can use (29) so that we can perform the sum in (44). This way we obtain the transformation formula

$${}_3\Phi_2 \left[ \begin{matrix} \omega, \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{matrix}; z \right] = A {}_3\Phi_2 \left[ \begin{matrix} \omega, z/\gamma_1, \beta_1/\alpha_1\gamma_1 \\ \omega/\gamma_1, \omega\alpha_2 z/\beta_2\gamma_1 \end{matrix}; \frac{\omega\alpha_1}{\beta_2} \right], \quad (45)$$

where the constant  $A$  can be written in several different forms, either with  ${}_2\Phi_1$ 's or with  $p(x)$ 's using (35). From this identity one can generate the symmetry relations of the cube in the Baxter–Bazhanov model under the 48 elements of the symmetry group of the cube, see also the recent work of Sergeev *et al.*<sup>43</sup>

In addition, one can work out relations for  ${}_4\Phi_3$  and higher. One of these many relations, *i.e.* a Saalschützian  ${}_4\Phi_3$  identity, is the star–triangle equation of the integrable chiral Potts model,<sup>14</sup> or its Fourier transform<sup>12</sup>

$$V_{pq}(a, b; n) \overline{W}_{qr}^{(f)}(n) = R_{pqr} V_{pqr}(a, b; n) W_{qr}(a - b). \quad (46)$$

More detail will be presented elsewhere. We also have to refer the reader to the recent work of Stroganov's group<sup>43</sup> which uses fairly different notations in their appendix. Their higher identities also involve Saalschützian cyclic hypergeometric functions, albeit that that is hard to recognize.

As a conclusion, we may safely state that the existence of all these cyclic hypergeometric identities is the mathematical reason behind the integrable chiral Potts family of models.

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